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NOTE ON A PROBLEM OF S. GOLDSTEIN

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NOTE ON A PROBLEM OF S. GOLDSTEIN.

§1 Introduction.

S. Goldstein considers the following problem.[1] To find a solution $f(r,\theta)$ of the differential equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \left(1 + \frac{1}{r^2}\right)\frac{\partial^2 f}{\partial \theta^2} = 0 \qquad \begin{matrix} 0 < r < \infty \\ -\theta_0 < \theta < \theta_0 \end{matrix} \tag{1}$$

which satisfies the boundary conditions

$$\theta = \pm\theta_0 \quad \begin{cases} \dfrac{\partial f}{\partial \theta} = -\dfrac{r^2}{1+r^2} & 0 < r < r_0 \\ f = 0 & r > r_0 \end{cases} \tag{2}$$

Goldstein's method is as follows. The regions $0 < r < r_0$ and $r_0 < r < \infty$ are taken separately and series expansions are derived which represent $f(r,\theta)$ in those regions. A number of unknown coefficients are introduced for which relations can be found by equating both expansions on the arc $r = r_0$. This method has a number of disadvantages, the main being the slow convergence of the iterative procedure by means of which the final solution is obtained.

In the present note alternative methods for obtaining the solution are offered in the hope that they may lead to a successful treatment of this problem and related problems with a smaller amount of numerical labour.

The following method makes use of a Green function of a related differential equation and reduces the problem to an integral equation which can be solved by an iterative procedure. This method can be summarized as follows.

A transformation can be found such that the (r,θ) region is transformed into the upper halfplane $v > 0$ of the u,v plane. The lines $\theta = \pm\theta_0$ are transformed into the positive and negative part of the line $v = 0$. Furthermore the transformation transforms (1) into

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = k(u,v)f \tag{3}$$

and the boundary conditions into

$$v=0 \qquad \begin{matrix} |u| < u_0 & \dfrac{\partial f}{\partial v} = \lambda(u) \\ |u| > u_0 & f = 0\,. \end{matrix} \tag{4}$$

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[1]) Proc. Royal Soc. A 123, p. 440-465 (1929).

Next a Green function is constructed satisfying

$$\frac{\partial^2 G}{\partial u^2} + \frac{\partial^2 G}{\partial v^2} = 0 \qquad (5)$$

$$v=0 \qquad |u| < u_0 \qquad \frac{\partial G}{\partial v} = 0$$

$$|u| > u_0 \qquad G = 0 \qquad (6)$$

$$G \sim -\tfrac{1}{2} \ln \left\{ (u-u')^2 + (v-v')^2 \right\}^{1/2} \text{ for } u \to u' \text{ and } v \to v'.$$

According to Green's theorem we have

$$2\pi f(u',v') + \iint_{v>0} (f \Delta G - G \Delta f) du\, dv = - \int_{v=0} \left(f\frac{\partial G}{\partial v} - G\frac{\partial f}{\partial v}\right) du$$

or

$$f(u',v') = \frac{1}{2\pi} \iint_{v>0} G(u,v,u',v')\, k(u,v)\, f(u,v) du\, dv +$$

$$+ \frac{1}{2\pi} \int_{-u_0}^{u_0} G(u,0,u',v')\, \lambda(u) du \qquad (7)$$

which is the required integral equation.

It turns out that G can be represented by elementary functions and that $k(u,v)$ is everywhere small except in the neighbourhood of $u = 0 \quad v = 0$ which corresponds to r small.

§ 2 Reduction to a Fredholm equation.

The following transformation

$$x = \sqrt{1+r^2} - \tfrac{1}{2} \ln \frac{\sqrt{1+r^2}+1}{\sqrt{1+r^2}-1} \qquad (8)$$

$$y = \theta$$

$$z = (1+r^2)^{1/4} f(r,\theta) \qquad (9)$$

transforms (1) into

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{r^2(4-r^2)}{4(1+r^2)^3} z \qquad (10)$$

and the boundary conditions into

$$y = \pm y_0 \begin{cases} \dfrac{\partial z}{\partial y} = - r^2(1+r^2)^{-3/4} & -\infty < x < x_0 \\ z = 0 & x > x_0, \end{cases} \qquad (11)$$

where x_0 and r are related according to (8).

Next a conformal transformation is applied by means of which the strip $|y| < y_0$ is mapped upon the upper halfplane $v > 0$ of the (u,v) plane. Take

$$u = - e^{\frac{\pi x}{2y_0}} \sin \frac{\pi y}{2y_0}$$
$$v = e^{\frac{\pi x}{2y_0}} \cos \frac{\pi y}{2y_0} \qquad (12)$$

or in complex notation

$$u + v i = i e^{\frac{\pi}{2y_0}(x+yi)}$$

The lines $y = \pm y_0$ are mapped upon the positive and negative part respectively of the line $v = 0$.
The differential equation (10) becomes

$$\Delta z = k(u,v)z \qquad (13)$$

where

$$k(u,v) = \frac{y_0^2}{\pi^2} \frac{1}{u^2+v^2} \frac{r^2(4-r^2)}{(1+r^2)^3} \qquad (14)$$

In (14) the old variable r still appears but by means of (8) and

$$x = \frac{y_0}{\pi} \ln(u^2+v^2) \qquad (15)$$

r can be expressed as a function of u^2+v^2.

The boundary condition (11) becomes

$$v=0 \begin{cases} -u_0 < u < u_0 & \frac{\partial z}{\partial v} = - \frac{2y_0 r^2}{\pi(1+r^2)^{3/4}} \cdot \frac{1}{u} \\ |u| > u_0 & z = 0 \end{cases} \qquad (16)$$

where r^2 can be expressed in u by means of

$$\ln u^2 = \frac{\pi}{y_0} \left\{ \sqrt{1+r^2} - \tfrac{1}{2} \ln \frac{\sqrt{1+r^2}+1}{\sqrt{1+r^2}-1} \right. . \qquad (17)$$

Next a Green function G(u,v,u',v') should be found satisfying

$$\Delta G = 0$$
$$v=0 \begin{cases} |u| < u_0 & \frac{\partial G}{\partial v} = 0 \\ |u| > u_0 & G = 0 \end{cases} \qquad (18)$$

and with $G(u,v,u',v') \sim - \frac{1}{2} \ln\left\{\sqrt{(u-u')^2+(v-v')^2}\right\}$ for $u \to u'$, $v \to v'$ simultaneously.

This function is

$$G(u,v,u',v') = \mathrm{Re} \int_{u+vi}^{\infty} \left[\frac{\{(u'+v'i)^2-u_0^2\}^{\frac{1}{2}}}{s-(u'+v'i)} - \frac{\{(u'-v'i)^2-u_0^2\}^{\frac{1}{2}}}{s-(u'-v'i)} \right] \frac{ds}{(s^2-u_0^2)^{\frac{1}{2}}} \tag{19}$$

We have

$$\int_{w}^{\infty} \frac{ds}{(s-a)\sqrt{s^2-b^2}} = \frac{1}{\sqrt{a^2-b^2}} \ln \left\{ \frac{(aw-b^2)+\sqrt{a^2-b^2}\sqrt{w^2-b^2}}{(w-a)(a+\sqrt{a^2-b^2})} \right\} \tag{20}$$

so that, if $w = u+vi$, $\overline{w} = u-vi$, $w' = u' + v'i$, $\overline{w}' = u' - v'i$

$$G(w,w') = \ln \left| \frac{ww'+(w^2-u_0^2)^{\frac{1}{2}}(w'^2-u_0^2)^{\frac{1}{2}}}{\overline{w}w'+(\overline{w}^2-u_0^2)^{\frac{1}{2}}(\overline{w}'^2-u_0^2)^{\frac{1}{2}}} \cdot \frac{\overline{w}-u_0}{w-u_0} \cdot \frac{\overline{w}'+(\overline{w}'^2-u_0^2)^{\frac{1}{2}}}{w'+(w'^2-u_0^2)^{\frac{1}{2}}} \right| \tag{21}$$

By means of Greens theorem the integral equation (7) is obtained with G defined by (21) and $k(u,v)$ by (14) and $\lambda(u)$ by (16).

§3 Reduction to a singular integral equation.

Next another method will be discussed.
Consider the system (1) and (2) and develop $f(r,\theta)$ in a Fourier series

$$f(r,\theta) = \sum_{1}^{\infty} c_n \sin \frac{n\pi\theta}{\theta_0} \tag{22}$$

By means of partial integration of the integral representation of c_n

$$c_n = \frac{1}{\theta_0} \int_{-\theta_0}^{\theta_0} f \sin \frac{n\pi\theta}{\theta_0} \, d\theta ,$$

the following relation is found

$$c_n = - \frac{2(-1)^n}{n\pi} f(r,\theta_0) - \frac{\theta_0}{n^2\pi^2} \int_{-\theta_0}^{\theta_0} \frac{\partial^2 f}{\partial \theta^2} \sin \frac{n\pi\theta}{\theta_0} \, d\theta . \tag{23}$$

On substituting

$$\frac{\partial^2 f}{\partial \theta^2} = - \frac{r}{1+r^2} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r}\right)$$

a non-homogeneous differential equation is obtained

$$\frac{d c_n^2}{dr^2} + \frac{1}{r}\frac{d c_n}{dr} - \omega^2\left(1 + \frac{1}{r^2}\right) c_n = \psi(r), \tag{24}$$

with

$$\omega = \frac{n\pi}{\theta_0}$$

and

$$\psi(r) = \frac{2(-1)^n n\pi}{\theta_0^2} \frac{1+r^2}{r^2} f(r,\theta_0).$$

The solution of the homogeneous equation is

$$c_n = A\, I_\omega(\omega r) + B\, K_\omega(\omega r) \; . \tag{25}$$

The Wronskian is

$$I K' - I' K = -\frac{1}{r} \; .$$

The solution of the non-homogeneous equation can also be represented by means of (25) but now A and B are functions of r with

$$\frac{dA}{dr} = r\, K_\omega(\omega r)\psi \; ,$$

$$\frac{dB}{dr} = -\, r\, I_\omega(\omega r)\psi \; .$$

This gives

$$c_n = -\frac{2(-1)^n}{n\pi} \int_0^\infty G_\omega(r,\varrho)\, f(\varrho,\theta_0)\, \frac{1+\varrho^2}{\varrho}\, d\varrho \; , \tag{26}$$

where

$$G_\omega(r,\varrho) = \begin{cases} \omega^2\, I_\omega(\omega r)\, K_\omega(\omega\varrho) & r < \varrho \\ \omega^2\, I_\omega(\omega\varrho)\, K_\omega(\omega r) & r > \varrho \end{cases} \tag{27}$$

For $\omega \to \infty$ we have the asymptotic expression

$$G_\omega(r,\varrho) \backsim \frac{\omega}{2(1+r^2)^{1/4}(1+\varrho^2)^{1/4}} \exp\left\{ -\omega \left| \ln \frac{r(1+\sqrt{1+\varrho^2})}{\varrho(1+\sqrt{1+r^2})} \right| - \right.$$

$$\left. -\omega \left| \sqrt{1+r^2} - \sqrt{1+\varrho^2} \right| \right\} \tag{28}$$

For $\omega \to \infty$ obviously $G_\omega \to 0$ except if $r = \varrho$.
We have always

$$\lim_{n \to \infty} \int_0^\infty G_\omega(r,\varrho)\, \frac{1+\varrho^2}{\varrho}\, d\varrho = 1 \; , \tag{29}$$

since for $r \backsim \varrho$

$$G_\omega(r,\varrho) \backsimeq \frac{\omega}{2(1+r^2)^{\frac{1}{2}}} \exp - \omega \frac{\sqrt{1+r^2}}{r} \,|\varrho - r| \; . \tag{30}$$

It is known that the series

$$\sum_1^\infty \left\{ c_n + \frac{2(-1)^n}{n\pi} f(r,\theta_0) \right\} \sin \frac{n\pi}{\theta_0}\theta = f(r,\theta) - \frac{\theta}{\theta_0} f(r,\theta_0) \tag{31}$$

is absolutely convergent and therefore can be differentiated. It is obtained

$$\sum_1 \left\{ n\pi c_n + 2(-1)^n f(r,\theta_0) \right\} \cos \frac{n\pi}{\theta_0}\theta = \theta_0 \frac{\partial f}{\partial \theta} - f(r,\theta_0) \; . \tag{32}$$

In this equation for $\theta = \theta_0$ the remaining boundary condition can be introduced

$$2\sum_{1}^{\infty}\left\{-(-1)^n \frac{n\pi}{2} c_n - f(r,\theta_0)\right\} = \frac{\theta_0 r^2}{1+r^2} + f(r,\theta_0). \tag{33}$$

Substitution of (26) gives

$$2\sum_{1}^{\infty}\left\{\int_0^{\infty} G_\omega(r,\varrho)\, f(\varrho,\theta_0)\frac{1+\varrho^2}{\varrho}\,d\varrho - f(r,\theta_0)\right\} = \frac{\theta_0 r^2}{1+r^2} + f(r,\theta_0). \tag{34}$$

Thus the problem is reduced to finding the function $f(r,\theta_0)$ from the integral equation (34) which is of a rather special kind. This equation might be solved by some method of successive approximation or iteration together with truncation of the series in (34).

In order to give an idea about the behaviour of (34) we take the following example

$$\sum_{1}^{\infty}\left\{\frac{n}{2}\int_{-\infty}^{\infty} e^{-n|t-\tau|} f(\tau)\,d\tau - f(t)\right\} = \lambda f(t) - g(t) \tag{35}$$

This can be written as

$$\sum_{1}^{\infty}\left\{\frac{n}{2}\int_0^{\infty} e^{-nu}\left\{f(t+u)+f(t-u)\right\}du - f(t)\right\} = \lambda f(t) - g(t)$$

or, after some reduction

$$\int_0^{\infty}\frac{1}{e^u+e^{-u}-2}\left\{\tfrac{1}{2} f(t+u) + \tfrac{1}{2} f(t-u) - f(t)\right\}du = \lambda f(t) - g(t)$$

This can be written in the form of a singular integral equation; for $\varepsilon \to 0$

$$\frac{1}{2}\int_{\varepsilon}^{\infty}\frac{f(t+u)+f(t-u)}{e^u+e^{-u}-2}\,du = \left(\lambda - \tfrac{1}{2} + \frac{1}{\varepsilon}\right) f(t) - g(t). \tag{36}$$

The equation (35) can be solved by means of a Laplace transformation. The integral in (35) is of the convolution type, and if $\varphi(s)$ and $\psi(s)$ are the transforms of $f(t)$ and $g(t)$ respectively we get

$$\sum_{1}^{\infty}\left\{\frac{n^2}{n^2-s^2}\varphi(s) - \varphi(s)\right\} = \lambda\varphi(s) - \psi(s)$$

or

$$\left(\tfrac{1}{2} - \tfrac{1}{2}\pi s \cot\pi s\right)\varphi(s) = \lambda\varphi(s) - \psi(s)$$

The Laplace transform of the solution of (35) is therefore

$$\varphi(s) = \frac{\psi(s)}{\tfrac{1}{2}\pi s\cot\pi s + \lambda - \tfrac{1}{2}} \tag{37}$$

If we consider the simple case $\lambda = \frac{1}{2}$, $g(t) = U(t)$, we find easily the back transform

$$f(t)=\begin{cases} 2 - \frac{2}{\pi^2} \sum_0^\infty (n+\frac{1}{2})^{-2} \; e^{-(n+\frac{1}{2})t} & t \geq 0 \\ \frac{2}{\pi^2} \sum_0^\infty (n+\frac{1}{2})^{-2} \; e^{(n+\frac{1}{2})t} & t \leq 0 \end{cases} \tag{38}$$

or

$$f(t)=\begin{cases} 2 - \frac{u}{\pi^2} \int_0^{e^{-t/2}} \frac{1}{u} \ln \frac{1+u}{1-u} \, du & t \geq 0 \\ \frac{u}{\pi^2} \int_0^{e^{t/2}} \frac{1}{u} \ln \frac{1+u}{1-u} \, du & t \leq 0 \end{cases} \tag{39}$$

4 Generalisations.

According to the theory of a screw propellor the velocity potential satisfies the potential equation

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{40}$$

The surface of the propellor is given by

$$s = \theta - \frac{z}{g(r)} \tag{41}$$

where $g(r)$ is a function of the radius which might be of the following type

$$g(r) = r^{-1/4} \left\{ c_0 r^4 - c_1 r^3 + c_2 r^2 - c_3 r + c_4 \right\} \quad . \tag{42}$$

The question arises whether it is possible to find a solution of (40) which is dependent on the two variables r and s only such that also the surface condition

$$\frac{\partial \phi}{\partial z} \cos\beta - \frac{\partial \phi}{\partial \theta} \frac{\sin\beta}{r} = \frac{c}{2} \cos\beta \, . \tag{43}$$

with

$$r \operatorname{tg}\beta = g(r)$$

is satisfied.
This means

$$\phi(r,z,\theta) = \phi\left(r, \theta - \frac{z}{g(r)}\right) \tag{44}$$

The answer is confirmative and without difficulty, we find

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \left(\frac{1}{r^2} + \frac{1}{g^2}\right) \frac{\partial^2 \phi}{\partial s^2} = 0 \tag{45}$$

and for $s \neq 0$

$$\frac{\partial \phi}{\partial s} = -\frac{c}{2} \frac{gr^2}{g^2+r^2} . \tag{46}$$

It is also possible to reduce (45) to the form (3).

If we write (45) in the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) + \frac{h^2(r)}{r^2}\frac{\partial^2 \phi}{\partial s^2} = 0 , \tag{47}$$

then the following transformation will be performed.

$$x = \int h \frac{dr}{r}$$

$$y = s \tag{48}$$

$$z = h^{\frac{1}{2}} \phi(r,s).$$

In view of

$$\frac{\partial f}{\partial r} = \frac{h}{r}\frac{\partial f}{\partial x}$$

and

$$\frac{\partial^2 f}{\partial r^2} = \frac{h^2}{r^2}\frac{\partial^2 f}{\partial x^2} + \left(\frac{h}{r}\right)' \frac{\partial f}{\partial x} ,$$

we find

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{d \ln h}{dx}\frac{\partial \phi}{\partial x} = 0 , \tag{49}$$

and next

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \left\{ h^{-\frac{1}{2}} (h^{-\frac{1}{2}})_{xx} + h^{-\frac{1}{2}} h_x (h^{-\frac{1}{2}})_x \right\} = 0 ,$$

or

$$\Delta z = \frac{1}{4}\frac{d}{dh}\left(\frac{h_x^2}{h}\right) z , \tag{50}$$

or

$$\Delta z = \frac{1}{2} r h^{-3/2} \frac{d}{dr}\left(r h^{-3/2} \frac{dh}{dr}\right) z . \tag{51}$$

A boundary condition (46) of the form

$$s=0 \qquad \frac{\partial \phi}{\partial s} = l(r) \tag{52}$$

is obviously transformed into

$$y=0 \qquad \frac{\partial z}{\partial y} = h^{\frac{1}{2}} l(r) . \tag{53}$$